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How to go with an RG Flow

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Abstract: We apply the formalism of holographic renormalization to domain wall solutions of 5-dimensional supergravity which are dual to deformed conformal field theories in 4 dimensions. We carefully compute one- and two-point functions of the energy-momentum tensor and the scalar operator mixing with it in two specific holographic flows, resolving previous difficulties with these correlation functions. As expected, two-point functions have a 0-mass dilaton pole for the Coulomb branch flow in which conformal symmetry is broken spontaneously but not for the flow dual to a mass deformation in which it is broken explicitly. A previous puzzle of the energy scale in the Coulomb branch flow is explained.

Keywords: AdS-CFT Correspondance, Renormalization Regularization and Renormalons, Brane Dynamics in Gauge Theories.

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1. Introduction

The holographic correspondence between (super)gravity theories on AdS spaces and (super)conformal field theories has passed many tests and generated much insight into the strong coupling behavior of field theory. For pure superconformal theories one can calculate many correlation functions from 5-dimensional supergravity, a procedure which is greatly facilitated by the high $\mathrm{SO}(4,2)$ symmetry of the bulk AdS_5 geometry. A number of domain wall solutions of supergravity have also been found — 5-dimensional geometries with the symmetry of the 4-dimensional Poincaré group — and general arguments show that these are dual to $d = 4$ superconformal theories deformed either by addition of relevant operators to the lagrangian or by vacuum expectation values of such operators. For a lagrangian deformation, conformal symmetry is explicitly broken and one expects that the trace of the stress tensor T_{ij} and the perturbing operator $\mathcal{O}$ are related by $T^i_i(x) = \beta_{\mathcal{O}}\mathcal{O}(x)$, where $\beta_{\mathcal{O}}$

is the beta function for the operator $\mathcal{O}$. For deformation by vacuum expectation value, conformal symmetry is spontaneously broken and one expects that $T_i^i = 0$. Correlation functions of the stress tensor should thus be a useful probe of the physics of holographic RG flows, yet there is a history of difficulty, briefly reviewed below, in attempts to calculate correlation functions which display the expected physics from supergravity.

The purpose of this paper is to outline how these difficulties are resolved by the formalism of holographic renormalization previously developed for AdS bulk geometry and various linear perturbations in AdS [1]. This formalism embodies the duality between UV divergences in the boundary field theory and IR divergences of the on-shell supergravity action. The IR divergences are determined by near boundary analysis of the classical supergravity equations of motion. The bulk theory is then regularized by adding counterterms, expressed as integrals of local expressions in the fields at boundary, to cancel these divergences. This procedure yields a finite renormalized action which is a functional of boundary data for the bulk metric and other bulk fields. In the AdS/CFT correspondence the boundary data are sources for dual operators and UV finite field theory correlation functions can be obtained by functional differentiation with respect to these sources. These correlators obey field theory Ward identities including conformal anomalies.

Near boundary analysis is sufficient to resolve all divergences, but leaves certain non-leading coefficients in the asymptotic expansion of solutions undetermined. These coefficients contain full information on the behavior of correlation functions at separated points in which much of the physics resides. To find them one needs a full solution of the equations of motion, usually specified uniquely by requiring that the solution vanish in the deep interior of the bulk geometry. A full solution of the nonlinear equations with general boundary data is far too difficult, but one can linearize about the background domain wall and, if fortunate, find explicit fluctuations which play the role of the bulk-to-boundary propagator in AdS geometries. Two-point correlation functions[1] can be found using these fluctuations. A simple cutoff method in momentum space [2, 3] is sufficient in many cases, but fails to give a full account of the stress tensor correlators in RG flows. We show how to determine these correlators from the renormalized action. Our formalism also simplifies the calculation of one-point functions.

Attention was first called to the stress tensor correlators in [4]. The linearized bulk field equations couple metric fluctuations to those of the scalar fields which flow in the domain wall backgrounds. It was shown how to decouple the equations for a general domain wall, and explicit fluctuations were found for two flows — the $GPPZ$ flow [5] describing a supersymmetric mass deformation of $\mathcal{N} = 4$ SYM

[1]In principle n-point correlators, for $n \geq 3$ can be calculated through Witten diagrams, but the integrals encountered are difficult in the reduced symmetry of domain walls. To our knowledge they have never been attempted, and we respect this tradition.

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theory and a Coulomb branch (CB) flow [6, 7] dual to spontaneous breaking of the gauge and conformal symmetry of $\mathcal{N} = 4$ SYM by a vacuum expectation value (vev) for the lowest chiral primary operator (CPO), bilinear in the fundamental scalars. Even with fluctuations known, the simple cutoff method failed to give physically reasonable two-point correlators. Progress was made in [8]. The main difference from [4] was a different choice of gauge for the bulk fluctuations which enabled the calculation of the correlation function $\langle T_i^i(x) T_j^j(y) \rangle$ for the $GPPZ$ flow. A gauge invariant formulation of the problem is clearly desirable, and this is incorporated in the method presented below.

There is other literature on fluctuations and correlation functions in the flows just described, mostly for uncoupled fluctuations such as transverse components of the bulk metric [9, 10, 11], or transverse bulk vectors dual to conserved currents [12]. The fluctuations of a number of other bulk fields, both coupled and uncoupled, were found in [13], in which the implications of supersymmetry and the multiplet structure of the fluctuations were emphasized. Correlators in Coulomb branch solutions of 10-dimensional Type IIB supergravity were discussed in [14, 15, 16].

The flows we deal with are supersymmetric. They are dual to boundary theories with unbroken $SUSY$, and $\langle T_{ij} \rangle = 0$ is thus required. In field theory calculation with a supersymmetric regulator automatically gives $\langle T_{ij} \rangle = 0$, but this relation would generically fail with a non-supersymmetric regulator. We find an analogous situation in our work in supergravity. The near boundary analysis used in the holographic renormalization procedure does not distinguish between $SUSY$ and non-$SUSY$ solutions of the field equations, so manifest supersymmetry is not guaranteed. On the other hand the renormalized action is ambiguous to the extent that a restricted class of finite local counterterms can be added. Requiring that the on-shell action evaluated on a bulk supersymmetric solution vanishes selects such a finite counterterm and $\langle T_{ij} \rangle = 0$ is then automatic.

Near boundary analysis of the field equations is straightforward but quite complicated and differs in detail from case to case depending on the dimension of the scalar field in the flow and its potential in the bulk action. We therefore try to be clear on the logical steps involved in the application of the holographic renormalization method to RG flows. But we simply present the asymptotic solutions of the field equations in an appendix A, with details of the procedure to be explained later [17].

The plan of the paper is as follows. In section 2 we review the construction of supersymmetric domain walls in $D = 5$. Our two examples can be lifted [6, 7, 18] to solutions of $D = 10$ Type IIB supergravity. We argue that a Weyl transformation should be made to a frame in which the AdS_5 scale becomes $L^2 = \alpha'$ (i.e. $L = 1$ in string units). This facilates comparison of energy scales in boundary and bulk and resolves a puzzle concerning the size of the mass gap in the Coulomb branch solution of [6, 7]. We also outline how the holographic formalism leads to the definition of the vevs $\langle T_{ij} \rangle$ and $\langle \mathcal{O} \rangle$ as functions of the sources. Higher point correlation functions can be obtained from these quantities, and we derive the important Ward and trace identities which they satisfy. In section 3 we explain how to use holographic renormalization to determine $\langle T_{ij}\rangle$ and $\langle \mathcal{O}\rangle$ explicitly in the two flows we study. Physical vevs are obtained at this stage of the program. In section 4 we give a general gauge invariant treatment of the linear fluctuation equations. In section 5 we present solutions of these equations and use our formalism to obtain two-point correlation functions. Their physical properties are discussed.

2. Holographic RG-flows

Holographic RG-flows are described by domain-wall spacetimes with scalar fields turned on. For one such active scalar with canonical kinetic term, the relevant part of the supergravity action is[2]

$$S = \int_M \mathrm{d}^5x\sqrt{G}\left[\frac{1}{4}R + \frac{1}{2}G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)\right] - \frac{1}{2}\int_{\partial M}\sqrt{\gamma}K\,, \tag{2.1}$$

where K is the trace of the second fundamental form. We work in Euclidean signature.

We are interested in domain wall solutions of the equations of motion (EOM's) resulting from (2.1) that preserve 4D Poincaré invariance. For one convenient choice of radial coordinate, they take the form

$$\begin{aligned} ds^2 &= e^{2A(r)}\delta_{ij}dx^i dx^j + dr^2 \\ \Phi &= \Phi(r)\,. \end{aligned} \tag{2.2}$$

The specific flows we consider are supersymmetric. From Killing spinor conditions in the bulk supergravity theory one can deduce the first order flow equations [19]

$$\frac{dA(r)}{dr} = -\frac{2}{3}W(\Phi)\,, \qquad \frac{d\Phi(r)}{dr} = \partial_\Phi W(\Phi)\,, \tag{2.3}$$

where $W(\Phi)$ is the superpotential. The potential $V(\Phi)$ is expressed in terms of W by

$$V(\Phi) = \frac{1}{2}(\partial_\Phi W)^2 - \frac{4}{3}W^2\,. \tag{2.4}$$

It is usually straightforward to solve the first order equations and any such solution automatically satisfies the second order equations of (2.1) for domain walls (but not conversely).

Supersymmetry guarantees that the domain wall solution is stable. However, the first order system can be derived from the requirement of gravitational stability

[2]Our curvature conventions are as follows $R_{\mu\nu\rho}{}^{\sigma} = \partial_\mu\Gamma_{\nu\rho}{}^{\sigma} + \Gamma_{\mu\lambda}{}^{\sigma}\Gamma_{\nu\rho}{}^{\lambda} - \mu \leftrightarrow \nu$ and $R_{\mu\nu} = R_{\mu\lambda\nu}{}^{\lambda}$. They differ by an overall sign from the conventions in [4, 13].

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of asymptotically AdS geometries even when the action (2.1) is not the truncation of a bulk supergravity theory. A generalized positive energy argument [20], was used in [21], and it was shown that $V(\Phi)$ must have the form (2.4) when there is a single scalar field.[3] The argument in [21] implies that there is a "superpotential" W such that the critical point of $V(\Phi)$ associated with the AdS geometry is also a critical point of W.[4] In the AdS/CFT correspondence, positivity of energy about a given AdS critical point is mapped into unitarity of the corresponding CFT. It follows that in all cases the dual CFT is unitary, the potential can be written as in (2.4) [23]. When (2.4) holds a simple BPS analysis [23, 22] of the domain wall action yields the flow equations (2.3).

We assume that $W(\Phi)$ has a stationary point at $\Phi = 0$. Near this point $W(\Phi)$ can be approximated by

$$W(\Phi) \approx -\left[\frac{3}{2} + \frac{\mu}{2}\Phi^2 + \mathcal{O}(\Phi^3)\right] \tag{2.5}$$

and we assume that $0 < \mu < 4$. For large positive r the domain wall solution is well approximated by the boundary region of an AdS_5 geometry with scale $L = 1$, i.e.

$$A(r) \approx r \qquad \Phi \approx \exp(-\mu r)\,. \tag{2.6}$$

If $0 < \mu < 2$, then the bulk field Φ is dual to an operator $\mathcal{O}$ of dimension $\Delta = 4 - \mu$ and the domain wall describes a relevant deformation of the CFT lagrangian. If $2 \leq \mu < 4$ then the dual operator has scale dimension $\Delta = \mu$ and the bulk flow describes spontaneous breaking by the vev $\langle\mathcal{O}\rangle$.[5 6]

In the discussion above we considered domain-wall solutions that asymptote to AdS_5 spacetimes with scale $L = 1$, rather than the official AdS/CFT scale $L = (4\pi\alpha'^2 g_s N)^{1/4}$. This is a significant point that we now discuss in some detail. As we review in the next sub-section, the on-shell gravitational action evaluated on the near-horizon solution is equated to the gauge theory effective action by the AdS/CFT

[3] For several scalars the obvious generalization of the form (2.4) implies stability, but the converse is not necesarily true.

[4] If one relaxes this requirement the potential can always be written in the form (2.4); one just views (2.4) as a differential equation for W [22]. In this case, however, as the original critical point may not be a critical point of W, the results of [21] about gravitational stability do not necesarily apply.

[5] For $\mu = \Delta = 2$ the scalar solution dual to a lagragian deformation is $\Phi \approx r\exp(-2r)$. Such a purely bosonic mass deformation breaks $SUSY$. Nevertheless one can obtain such solutions from the first order equations. The relevant "superpotential" is given by $W(\Phi(r)) = -3/2 - \exp(-4r)(1/8 - r/2 + r^2)$. This "superpotential" is not analytic in Φ (to obtain $W(\Phi)$ one needs to invert $\Phi = r\exp(-2r)$).

[6] A more detailed argument [24] shows that operators of dimension $1 \leq \mu < 2$ can be described holographically.

correspondence. The latter has a large N expansion of the form

$$S_{\text{gauge}} = \sum_{g=0}^{\infty} N^{2-2g} \sum_{k=0}^{\infty} c_{g,k} (g_{YM}^2 N)^k \,. \tag{2.7}$$

On the gravitational side, specifically IIB string theory in the usual Einstein frame, the overall constant in front of the action is $1/16\pi G_N^{(10)}$, where

$$G_N^{(10)} = 8\pi^6 g_s^2 \alpha'^4 \tag{2.8}$$

is the $D = 10$ Newton's constant. So the gravitational action seems at first sight to have a different leading behavior. As is well known the correct dependence on N and $4\pi g_s = g_{YM}^2$ is restored because the near-horizon $D3$-brane metric depends on these quantities. To provide a more manifest match to the gauge theory expansion we perform a constant Weyl rescaling [25] so that the solution now becomes asymptotic to $AdS_5 \times S^5$ with AdS and sphere radius equal to one in string units, i.e. $L^2 = \alpha'$. Newton's constant is now equal to

$$G_N^{(10)} = 8\pi^6 \frac{\alpha'^4}{16\pi^2 N^2} = 8\pi^6 g_s^2 \left(\frac{\alpha'}{\sqrt{4\pi g_s N}} \right)^4 \,. \tag{2.9}$$

The five dimensional Newton's constant $G_N^{(5)}$ is now obtained by dividing (2.9) by the volume of the unit five-sphere, $vol(S^5) = \pi^3$. It follows that the overall constant in front of the five dimensional action (2.1) is $N^2/2\pi^2$. The difference between (2.8) and (2.9) means that energies are measured in different units, and the effective α' in the two frames differ by a factor of $\sqrt{g_s N}$. This is the origin of the factor $\sqrt{g_s N}$ present in the UV/IR relation derived in [26].

The Weyl scaling needed is the special case ($p = 3$) of the rescaling used to reach the so-called "dual-frame" [25].[7] In this frame all Dp-branes,[8] not just the $D3$-brane, have a near-horizon limit $AdS_{p+2} \times S^{8-p}$. The Weyl rescaling and the dilaton are not constant when $p \neq 3$ and because of this the solutions are $1/2$ supersymmetric even in the near-horizon limit. It has been argued in [25] that this frame is holographic. It is in this frame that the on-shell gravitational action has manifestly the same leading behavior at large N as the gauge theory effective action, and the supergravity variables automatically take into account the UV/IR relation. So in order to have manifest matching between gauge theory and supergravity computations (without worrying about different units) we need to do the gravitational computation in the dual-frame. For the case at hand this simply means that we will consider the solution with $L = 1$ (we set $\alpha' = 1$ from now on), and Newton's constant is equal to $1/N^2$.

[7] The dual-Dp-frame is defined as the frame where the metric and the $8-p$-field strength couple to the dilaton the same way.

[8] Fivebranes are exceptional in that the near-horizon limit gives $Mink_7 \times S^3$ with a linear dilaton.

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2.1 Correlators and Ward identities

To obtain correlation functions we must go back to the second order EOM's of (2.1) and consider solutions with arbitrary Dirichlet boundary conditions for the bulk fields. As in earlier papers on holographic renormalization, we use a different radial variable $\rho = \exp(-2r)$, where the boundary is mapped to $\rho = 0$. The bulk scalar satisfies the (modified) Dirichlet condition

$$\Phi(\rho, x) \to \rho^{\frac{4-\Delta}{2}} \phi_{(0)}(x) \quad (\text{as } \rho \to 0)\,. \tag{2.10}$$

The general bulk metric ansatz is

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j\,, \tag{2.11}$$

where the boundary metric limits to

$$g_{ij}(x,\rho) \to g_{(0)ij}(x) \quad (\text{as } \rho \to 0)\,. \tag{2.12}$$

In the AdS/CFT correspondence, the boundary data $g_{(0)ij}(x)$ and $\phi_{(0)}(x)$ are arbitrary functions of the transverse coordinates x^i and are the sources for the stress tensor T_{ij} and, respectively, the operator $\mathcal{O}$ in the boundary field theory. The correspondence is then expressed by the basic formula for the generating functional

$$\langle \exp(-S_{QFT}[g_{(0)}] - \int d^4x \sqrt{g_{(0)}} \mathcal{O}(x) \phi_{(0)}(x)) \rangle = \exp(-S_{SG}[g_{(0)}, \phi_{(0)}])\,. \tag{2.13}$$

On the left side $\langle \ldots \rangle$ denotes the functional integral average involving the field theory action $S_{QFT}[g_{(0)}]$ minimally coupled to $g_{(0)}$. The action $S_{SG}[g_{(0)}, \phi_{(0)}]$ is the classical action integral of (2.1) evaluated on the classical solution with boundary data $g_{(0)}$ and $\phi_{(0)}$. This is the leading term in the semiclassical computation of the supergravity partition function. Unless confusion arises we will henceforth use S to denote this classical on-shell action.

Actually S is divergent due to the behavior of the solution $g_{ij}(\rho,x), \Phi(\rho,x)$ near the AdS boundary. One must regularize the action, e.g. by cutting off the radial integration at a small value $\rho = \epsilon$, and add appropriate boundary counterterms $S_{\rm ct}$ to cancel the divergences. The action S must then be replaced by the renormalized $S_{\rm ren} = \lim_{\epsilon\to 0}(S_{\rm reg} + S_{\rm ct})$ in (2.13) and $S_{\rm ren}$ becomes the generating functional of connected correlation functions. This process is the heart and soul of holographic renormalization and is described in the next section. In the rest of this section we discuss some general properties of the correlation functions obtained from the procedure.

By definition, the variation of the effective action is equal to[9]

$$\delta S_{\rm ren}[g_{(0)ij}, \phi_{(0)}] = \int d^4x \sqrt{g_{(0)}} \left[\frac{1}{2} \langle T_{ij} \rangle \delta g^{ij}_{(0)} + \langle \mathcal{O} \rangle \delta \phi_{(0)} \right]. \tag{2.14}$$

[9]Notice that the definition of $\langle \mathcal{O} \rangle$ differs by a sign from the definition used in [1].

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The expectation values $\langle T_{ij}\rangle$ and $\langle\mathcal{O}\rangle$ are functions of the sources to be computed in the next section. Multi-point correlation functions can be obtained by further differentiation, e.g.

$$\langle T_{ij}(x)T_{kl}(y)\rangle = -\frac{2}{\sqrt{g_{(0)}(y)}}\frac{\delta\langle T_{ij}(x)\rangle}{\delta g^{kl}_{(0)}(y)}\,. \tag{2.15}$$

A well known feature of AdS/CFT physics is the correspondence between bulk gauge symmetries and global symmetries of the boundary theory. In the present setting the relevant gauge symmetries are bulk diffeomorphisms. Some of these were used to bring the bulk metric to the form (2.11). The remainder preserve the form (2.11), and we distinguish between diffeomorphisms involving only the 4 transverse coordinates:

$$\delta g^{ij}_{(0)} = -(\nabla^i\xi^j + \nabla^j\xi^i), \qquad \delta\phi_{(0)} = \xi^i\nabla_i\phi_{(0)} \tag{2.16}$$

and a one-parameter subalgebra of $5D$ diffeos [27] whose effect on the boundary data coincides with the Weyl transformation

$$\delta g^{ij}_{(0)} = -2\sigma g^{ij}_{(0)}, \qquad \delta\phi_{(0)} = (\Delta-4)\sigma\phi_{(0)}\,. \tag{2.17}$$

The counterterms needed to make the on-shell action finite are manifestly invariant under the $4D$ diffeomorphisms, but they break the diffeos that induce Weyl transformations on the boundary [28]. It follows that the diffeomorphism Ward identity should hold, but the trace Ward identity is expected to be broken.

It is straightforward to substitute the variations (2.16)–(2.17) in (2.14) and obtain the Ward and trace identities[10]

$$\nabla^i\langle T_{ij}\rangle = -\langle\mathcal{O}\rangle\nabla_j\phi_{(0)} \tag{2.18}$$

$$\langle T^i_i\rangle = (\Delta-4)\phi_{(0)}\langle\mathcal{O}\rangle + \mathcal{A}\,, \tag{2.19}$$

where $\mathcal{A}$ is the conformal anomaly. As mentioned above, it originates from the fact that the counterterms break part of the $5D$ diffeomorphisms. The explicit form of $\mathcal{A}$ will be determined below. It is useful to note here that $\mathcal{A}$ is local in the sources. One of the most elegant aspects of the holographic renormalization formalism is the simple emergence of Ward and trace identities including conformal anomalies.

The source for the stress energy tensor can be decomposed as follows,

$$\delta g_{(0)ij} = \delta h^T_{(0)ij} + \nabla_{(i}\delta h^L_{(0)j)} + \delta_{ij}\frac{1}{4}\delta h_{(0)} - \nabla_i\nabla_j\delta H_{(0)}\,, \tag{2.20}$$

[10] Since $\langle T_{ij}\rangle$ includes the scalar source term in (2.13) these identities do not quite have the standard field theory form. For this one must use $\langle T_{ij}\rangle_{QFT} = \langle T_{ij}\rangle + \varphi_{(0)}\langle\mathcal{O}\rangle g_{(0)ij}$, where $\varphi_{(0)}$ is the fluctuation part of the source, see footnote 12 for notation. Notice that a possible background part of the source is the coupling constant of an operator deformation and $\langle T_{ij}\rangle_{QFT}$ denotes the stress energy tensor of the deformed CFT, not the original CFT.

where

$$\nabla^i h^T_{(0)ij} = 0\,, \qquad h^{T\ i}_{(0)i} = 0\,, \qquad \nabla^i h^L_{(0)i} = 0\,. \tag{2.21}$$

All covariant derivatives are that of $g_{(0)}$.

Using (2.20) and (2.18)–(2.19) and partial integration we can rewrite (2.14) as

$$\begin{aligned}\delta S_{\rm ren} = \int \mathrm{d}^4x\sqrt{g_{(0)}}\Big(&-\frac{1}{2}\delta h^{T\ ij}_{(0)}\langle T_{ij}\rangle - \frac{1}{2}\delta h^{L\ i}_{(0)}\langle\mathcal{O}\rangle\nabla_i\phi_{(0)} - \\ &-\frac{1}{2}\delta H_{(0)}[\nabla^i\langle O\rangle\nabla_i\phi_{(0)} + \langle\mathcal{O}\rangle\nabla^2\phi_{(0)}] - \\ &-\frac{1}{8}\delta h_{(0)}[(\Delta-4)\phi_{(0)}\langle\mathcal{O}\rangle + \mathcal{A}] + \delta\phi_{(0)}\langle\mathcal{O}\rangle\Big)\,. \end{aligned}\tag{2.22}$$

The form above is quite general, but we now make two assumptions appropriate to our situation. We assume that the sources describe an x-independent domain wall and linear fluctuations above it. This is sufficient to study one- and two-point functions. In this case either $\nabla^i\langle\mathcal{O}\rangle_B = 0$ or $\nabla_i\phi_{(0)B} = 0$, where the sub-index denotes a background value. The term proportional to $\delta h^{L\ i}_{(0)}$ vanishes because either the source or the vev is constant and in the latter case the term drops out upon partial integration. One then finds

$$\begin{aligned}\delta S_{\rm ren} = \int \mathrm{d}^4x\sqrt{g_{(0)}}\Big[&-\frac{1}{2}\delta h^{T\ ij}_{(0)}\langle T_{ij}\rangle - \frac{1}{2}\delta H_{(0)}\langle\mathcal{O}\rangle\nabla^2\phi_{(0)} + \\ &+\frac{1}{8}\delta h_{(0)}[(4-\Delta)\phi_{(0)}\langle\mathcal{O}\rangle - \mathcal{A}] + \delta\phi_{(0)}\langle\mathcal{O}\rangle\Big]\,. \end{aligned}\tag{2.23}$$

Since $\langle T^i_i\rangle = -8\delta S_{\rm ren}/\delta h_{(0)}$ the expression in (2.23) shows that all correlation functions of T^i_i and $\mathcal{O}$ can be obtained from the form of $\langle\mathcal{O}\rangle$ and $\mathcal{A}$ as functions of the sources. This is a consequence of the trace identity (2.19).

The two-point function of T_{ij} has the standard representation

$$\langle T_{ij}(p)T_{kl}(-p)\rangle = \P^{TT}_{ijkl}A(p^2) + \pi_{ij}\pi_{kl}B(p^2) \tag{2.24}$$

in terms of the projection operators $\pi_{ij} = \delta_{ij} - p_ip_j/p^2$ and

$$\P^{TT}_{ijkl} \equiv -\frac{\delta h^T_{(0)ij}}{\delta h^{T\ kl}_{(0)}} = \frac{1}{2}(\pi_{ik}\pi_{jl} + \pi_{il}\pi_{jk}) - \frac{1}{3}\pi_{ij}\pi_{kl} \quad . \tag{2.25}$$

The transverse traceless (TT) amplitude $A(p^2)$ can thus be calculated by further variation of the TT projection of $\langle T_{ij}\rangle$, while $B(p^2) = \langle T^i_i(p)T^j_j(-p)\rangle/9$. The Ward identity implies that $\langle T_{ij}(p)\mathcal{O}(-p)\rangle = \pi_{ij}C(p^2)$ with invariant amplitude $C(p^2) = \langle T^i_i(p)\mathcal{O}(-p)\rangle/3$. Note that $\langle T_{ij}(p)\mathcal{O}(-p)\rangle$ is the connected correlator. When the background vev $\langle\mathcal{O}\rangle$ does not vanish one must correct for the source term in Footnote 10 to obtain this. We will obtain these correlators, together with $\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle$ for the CB and $GPPZ$ flows in section 5.

3. Holographic renormalization

We will use the renormalization method pioneered in [28] and developed in [1], see also [29] for related work. In this method one regulates the divergent on shell action S by restricting the ρ integration to $\rho \geq \epsilon$. Asymptotic solutions of the field equations with arbitrary Dirichlet boundary data (= field theory sources) are then obtained and used to express the divergences as $\epsilon \to 0$ in terms of the sources. Finally one adds counterterms to cancel these divergences. Here we will only give the results; details will be presented elsewhere.

The solutions we consider are asymptotically AdS. This means that near the boundary one can find coordinates such that the metric $g_{ij}(x,\rho)$ in (2.11) can be expanded in a series of the form

$$g(x,\rho) = g_{(0)} + g_{(2)}\rho + \rho^2[g_{(4)} + h_{1(4)}\log\rho + h_{2(4)}\log^2\rho] + \dots \tag{3.1}$$

The scalar field can also be expanded in a similar fashion. Since the exact form of the expansion depends on the mass of the bulk scalar, it will be presented below for the two cases we consider.

The next step is the near boundary analysis of the EOM's. In this process one subtitutes the assumed expansions into the EOM's and solves them iteratively. In this way many higher order terms in the expansion are determined as (local) functions of the sources, but not all terms are so determined. For the metric, $g_{(4)}$ is the first term which is not fully determined (although its trace and covariant divergence are determined). It is to be expected that near boundary analysis does not completely fix the solution of second order field equations with a Dirichlet condition on an AdS boundary. Additional information on the behavior in the deep interior of the space-time is required. The product of this phase of the procedure is an asymptotic solution of the EOM's in which the unspecified coefficients are simply carried as such within the series expansions.

It turns out that divergences of the cutoff action are fully determined in terms of the sources by near boundary analysis. The divergences can be expressed as counter terms which are boundary integrals of local invariants constructed from the induced metric $\gamma_{ij} = \frac{1}{\epsilon}g_{(0)ij}(x)$ and the scalar field $\Phi(x,\epsilon)$. One adds these counter terms to define the finite renormalized action $S_{\text{ren}} = S_{\text{reg}} + S_{\text{ct}}$ in which the limit $\epsilon \to 0$ may be taken.

The exact form of the counterterms depends on the specific potential of the scalar field. For a given potential, however, the derived counterterms are universal, i.e. the on-shell action will be finite for *any* solution of the bulk field equations. This is a property that any holographic renormalization scheme should have.

The renormalization procedure is ambiguous to the extent that finite local counter terms can be added to S_{ren}. This corresponds to scheme dependence in quantum field theory. In particular, S_{ren} need not incorporate requirements of super-

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symmetry, since it was derived using counterterms valid for the most general non-supersymmetric solution of the EOM's. The particular requirement which need not be satisfied is that $E_{vac} = 0$ in a supersymmetric vacuum. In holography this means that $S_{\rm ren} = 0$ when evaluated in the background geometry of a solution of (2.3). We will use a supersymmetric scheme to fix the form of the counterterms.

It is much easier to compute the regularized on-shell action for domain wall solutions (2.2) than for those with x-dependent boundary data. For solutions of (2.3) the answer can be read off from the BPS form of the action in [23, 22] (see (13) in [23] or (14) in [22]). One obtains

$$S_{\rm bkgd,reg} = \int_{\rho=\epsilon} d^4x\sqrt{\gamma}W[\Phi]\,, \tag{3.2}$$

where $\gamma_{ij} = e^{2A}\delta_{ij}$ is the induced metric at the cutoff surface $\rho = \epsilon$ (or equivalently $r = -\frac{1}{2}\log\epsilon$). When $W(\Phi)$ is expressed as a series, as in (2.5), the low order terms will be divergent and these must agree with similar terms obtained from the divergences of more general solutions. In addition, depending on the asymptotic behaviour in ρ of the scalar field, there may be a residual finite part. If present this must be subtracted for the scheme to be supersymmetric, ensuring that $S_{\rm ren} = 0$ in the background. The specific working of this mechanism will be discussed with our examples below.

It is both interesting and helpful that one can determine some divergences of the general $S_{\rm reg}$ by examining simple subclasses of solutions of the theory, in our case SUSY domain wall solutions. However, one must study more general solutions in order to obtain conterterms necessary to cancel divergences in all correlation functions. For example, there are counterterms involving $\partial_i\phi_{(0)}$ and $\partial_i g_{(0)jk}$, which cannot be found using x-independent domain wall solutions.

3.1 Coulomb branch

Our first example is the supergravity dual of a particular state in the Coulomb branch of $\mathcal{N} = 4$ SYM theory. It can be obtained by turning on the SO(4) $\times$ SO(2) singlet component of the scalar field dual to CPO in the $\mathbf{20}'$ of SO(6). We henceforth denote the active scalar field by Φ.

The superpotential is given by

$$W(\Phi) = -e^{-\frac{2\Phi}{\sqrt{6}}} - \frac{1}{2}e^{\frac{4\Phi}{\sqrt{6}}} = -\frac{3}{2} - \Phi^2 + \mathcal{O}(\Phi^3)\,. \tag{3.3}$$

The domain-wall solution is given by

$$v = e^{\sqrt{6}\Phi}, \qquad e^{2A} = l^2\frac{v^{2/3}}{1-v}, \qquad \frac{dv}{dr} = 2v^{2/3}(1-v)\,. \tag{3.4}$$

The boundary is at $v = 1$. There is a curvature singularity at $v = 0$ which is the origin of a disc distribution of $D3$-branes in Type-IIB supergravity. See [6, 7, 13, 12] for more details of this solution and previously studied correlation functions.

The change of variables that brings the domain-wall metric to the coordinate system (2.11) is given by

$$1 - v = l^2\rho - \frac{2}{3}l^4\rho^2 + \mathcal{O}(\rho^3)\,. \tag{3.5}$$

In these coordinates the solution φ_B, A, is given by

$$\varphi_B = \frac{1}{\sqrt{6}}\left(-\rho l^2 + \frac{1}{6}l^4\rho^2 + \mathcal{O}(\rho^3)\right), \qquad e^{2A} = \frac{1}{\rho}\left(1 - \frac{1}{18}l^4\rho^2 + \mathcal{O}(\rho^3)\right). \tag{3.6}$$

By inspection (3.2) evaluated on the background solution (3.6) is divergent. It follows from the given asymptotics that only the first two terms in the expansion of W around $\Phi = 0$ contribute to the IR divergences, and that there is no finite term. All the other terms in the expansion of W vanish in the limit $\epsilon \to 0$. Thus the counterterms needed to make the background action finite are given by

$$S_{\mathrm{ct,bkgd}} = \int_{\rho=\epsilon} \mathrm{d}^4x\sqrt{\gamma}\left(\frac{3}{2} + \Phi^2\right). \tag{3.7}$$

We will see that these are part of the counterterms required to make finite the on-shell action in general. Since there is no finite term left after the subtraction, the renormalized action evaluated on the background solution equals zero, as required by supersymmetry.

To obtain the general form of the counterterms we note that the asymptotic expansion for a bulk scalar field of AdS mass $m^2 = -4$ dual to an operator of conformal dimension $\Delta = 2$ reads,

$$\Phi(x,\rho) = \rho\log\rho(\phi_{(0)} + \phi_{(2)}\rho + \rho\log\rho\psi_{(2)} + \cdots) + \rho(\tilde{\phi}_{(0)} + \cdots)\,. \tag{3.8}$$

Inserting the asymptotic expansions in the bulk field equations one finds that the coefficients shown in (3.1) are uniquely determined in terms of $g_{(0)}$ and $\phi_{(0)}$ except for $g_{(4)ij}$ that is only partially determined and $\tilde{\phi}_{(0)}$ that is undetermined. In particular, only $\mathrm{Tr}g_{(4)}$ and $\nabla^i g_{(4)ij}$ are determined. The exact expressions are given in the appendix. The coefficients $g_{(4)ij}$ and $\tilde{\phi}_{(0)}$ are related to the holographic one-point functions in the presence of sources [1] as we will derive shortly.

Knowledge of the asymptotic solution allows one to evaluate the regularized action and obtain the divergences. These can be cancelled by adding the following covariant counterterms

$$S_{\mathrm{ct}} = \int_{\rho=\epsilon} \mathrm{d}^4x\sqrt{\gamma}\left(\left[\frac{3}{2} - \frac{1}{8}R - \frac{1}{32}\log\epsilon(R_{ij}R^{ij} - \frac{1}{3}R^2)\right] + \left[\Phi^2(x,\epsilon) + \frac{\Phi^2(x,\epsilon)}{\log\epsilon}\right]\right), \tag{3.9}$$

where γ is the induced metric at $\rho = \epsilon$. All curvatures are of the induced metric. Notice that the term $\Phi^2/\log\epsilon$ is divergent in the limit $\epsilon \to 0$ with $\phi_{(0)}$ fixed (but equal to zero when $\phi_{(0)} = 0$ which is the case for the SUSY domain wall solutions). Thus the set of counterterms contains (3.7) and more.

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The renormalized action is equal to

$$S_{\rm ren} = \lim_{\epsilon\to 0} \left(S_{\rm reg} + S_{\rm ct}\right), \tag{3.10}$$

where $S_{\rm reg}$ is the on-shell action in (2.1) regulated by restricting the range of integration to $\rho \geq \epsilon$.

The expectation value of the operator dual to Φ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\rm ren}}{\delta \phi_{(0)}} = \lim_{\epsilon\to 0} \left(\frac{\log \epsilon}{\epsilon} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{\rm ren}}{\delta \Phi(x,\epsilon)} \right). \tag{3.11}$$

By straightforward computation of the variational derivative [17] one obtains

$$\langle \mathcal{O} \rangle = 2\tilde{\phi}_{(0)}\,. \tag{3.12}$$

One can similarly compute [17] the expectation value of the stress-energy tensor. By definition

$$\langle T_{ij} \rangle = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_{\rm ren}}{\delta g^{ij}_{(0)}} = \lim_{\epsilon\to 0} \left(\frac{1}{\epsilon} \frac{2}{\sqrt{\gamma(x,\epsilon)}} \frac{\delta S_{\rm ren}}{\delta \gamma^{ij}(x,\epsilon)} \right). \tag{3.13}$$

After some computation one finds that all infinities cancel and the finite part is equal to

$$\begin{aligned}\langle T_{ij} \rangle = {} & g_{(4)ij} + \frac{1}{8}\left[\operatorname{Tr} g_{(2)}^2 - (\operatorname{Tr} g_{(2)})^2\right] g_{(0)ij} - \frac{1}{2}(g_{(2)}^2)_{ij} + \frac{1}{4} g_{(2)ij} \operatorname{Tr} g_{(2)} + \\ & + \frac{1}{3}(\tilde{\phi}_{(0)}^2 - 3\phi_{(0)}\tilde{\phi}_{(0)}) g_{(0)ij} + \frac{2}{3}\phi_{(0)}^2 g_{(0)ij} + \frac{3}{2} h_{(4)ij}\,.\end{aligned} \tag{3.14}$$

The last two terms can be cancelled by adding finite local counterterms to the action. The last term is proportional to the stress energy tensor derived from an action given by the gravitational conformal anomaly (3.22) [1]. The next-to-last term is proportional to the stress energy tensor derived from an action equal to the matter conformal anomaly (3.23).

The computation of $\langle \mathcal{O} \rangle$ and $\langle T_{ij} \rangle$ was carried out in a specific coordinate system. One may wonder whether these results are sensitive to our specific choice. From the derivation of $\langle \mathcal{O} \rangle$ and $\langle T_{ij} \rangle$ as functional derivatives of $S_{\rm ren}$ it follows that they transform as tensors up to the contribution of the conformal anomaly. The contribution of the conformal anomaly to the transformation rules is also straightfoward to obtain in all generality [1, 30]. Here we only discuss how $\langle \mathcal{O} \rangle$ and $\langle T_{ij} \rangle$ transform under a constant rescaling of the ρ variable. This is of particular interest because the much-discussed association of the bulk radial coordinate with energy scale in the boundary field theory suggests that the transformation

$$\rho = \rho' \mu^2\,, \qquad x^i = x^{i\prime} \mu \tag{3.15}$$

introduces the RG scale μ. This transformation is an isometry of AdS spacetime (with a metric given by (2.11) with $g_{(0)ij} = \delta_{ij}$ and all other $g_{(k)ij} = 0$). Under this transformation most coefficients pick up overall factors of μ according to their dimension, but there are also non-trivial transformations due to the logarithms in the asympotic solutions. One obtains

$$\begin{aligned} \phi'_{(0)}(x') &= \mu^2\phi_{(0)}(x'\mu)\,, \qquad g'_{(0)}(x') = g_{(0)}(x'\mu)\,, \qquad g'_{(2)}(x') = \mu^2 g_{(2)}(x'\mu) \\ \tilde{\phi}'_{(0)}(x') &= \mu^2[\tilde{\phi}_{(0)}(x'\mu) + \log\mu^2\phi_{(0)}(x'\mu)]\,, \\ g'_{(4)}(x') &= \mu^4\left[g_{(4)} + \log\mu^2\left(h_{(4)} - \frac{2}{3}\phi_{(0)}\tilde{\phi}_{(0)}g_{(0)}\right) - (\log\mu^2)^2\frac{1}{3}\phi^2_{(0)}g_{(0)}\right](x'\mu)\,. \end{aligned} \tag{3.16}$$

It follows that

$$\begin{aligned} \langle\mathcal{O}(x')\rangle' &= \mu^2\left(\langle\mathcal{O}(x'\mu)\rangle + \log\mu^2 2\phi_{(0)}(x'\mu)\right) \\ \langle T_{ij}(x')\rangle' &= \mu^4\left(\langle T_{ij}(x'\mu)\rangle + \log\mu^2[h_{(4)ij} - \phi^2_{(0)}g_{(0)ij}](x'\mu)\right). \end{aligned} \tag{3.17}$$

It is satisfying to see that the new terms can be obtained from the following local finite counterterm,

$$S_{\text{fin}}(\mu) = \int \mathrm{d}^4x\sqrt{g_{(0)}}\log\mu^2\frac{1}{2}\mathcal{A}\,, \tag{3.18}$$

where $\mathcal{A} = \mathcal{A}_{grav} + \mathcal{A}_{scal}$ is the conformal anomaly given in (3.22) and (3.23). This implies the transformation (3.15) only adds contact terms to correlation functions, and scales the momenta by $1/\mu$.

From the expressions in (3.12) and (3.14) we find the vevs with all sources equal to zero are given by

$$\langle\mathcal{O}\rangle_B = -\frac{2}{\sqrt{6}}\frac{N^2}{2\pi^2}l^2, \qquad \langle T_{ij}\rangle_B = 0\,. \tag{3.19}$$

The term $N^2/2\pi^2$ is the overall constant in front of the action[11] discussed in section 2. The vevs show that the solution describes a state in the moduli space of vacua, as promised. As one might have expected the size of the vev, l^2, is set by the size of the symmetry breaking effect; the radius of the disk distribution of D3-branes.

It is straightforward to use the solution of the bulk field equations given in appendix A.1 to show that

$$\nabla^i\langle T_{ij}\rangle = -\langle\mathcal{O}\rangle\nabla_j\phi_{(0)} \tag{3.20}$$

$$\langle T^i_i\rangle = -2\phi_{(0)}\langle\mathcal{O}\rangle + \frac{1}{16}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) + 2\phi^2_{(0)} \tag{3.21}$$

[11] Here and henceforth we adopt the policy of including this factor only when final results for correlation functions are given.

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i.e. $\langle T_{ij} \rangle$ correctly satisfies the diffeomorphism and trace Ward identities. The last two terms in (3.21) are what we called $\mathcal{A}$ in (2.19). The second term

$$\mathcal{A}_{\text{grav}} = \frac{1}{16}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) \tag{3.22}$$

is the holographic gravitational conformal anomaly [28] and the last term

$$\mathcal{A}_{\text{scal}} = 2\phi_{(0)}^2 \tag{3.23}$$

is the conformal anomaly due to matter [31]. The coefficients in both of them are known not to renormalize, and indeed we obtain the correct value. The Ward identities and the anomalies are important checks of the intermediate computations and of the consistency of the formalism.

3.2 *GPPZ* flow

Our second example is the supergravity dual of a $\mathcal{N} = 1$ supersymmetry preserving mass deformation of $\mathcal{N} = 4$ SYM theory [5]. We will consider only the simplest case in which the active scalar field is one of the two SO(3) singlet, dimension $\Delta = 3$ scalars studied in [5]. Specifically we consider the field called m, here renamed Φ, which is dual to a chiral fermion mass operator, and we do not treat a more general flow involving m and the second scalar σ.

The superpotential reads

$$W(\Phi) = -\frac{3}{4}\left[1 + \cosh\left(\frac{2\Phi}{\sqrt{3}}\right)\right] = -\frac{3}{2} - \frac{1}{2}\Phi^2 - \frac{1}{18}\Phi^4 + \mathcal{O}(\Phi^5)\,. \tag{3.24}$$

The domain-wall solution is given by

$$\varphi_B = \frac{\sqrt{3}}{2}\log\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\,, \qquad e^{2A} = \frac{u}{1-u}\,, \qquad \frac{du}{dr} = 2(1-u)\,. \tag{3.25}$$

The u variable is related to the ρ variable by $u = 1 - \rho$. Near the boundary the solution has the expansion

$$\varphi_B = \rho^{1/2}\left[\sqrt{3} + \rho\frac{1}{\sqrt{3}} + \mathcal{O}(\rho^2)\right], \qquad e^{2A} = \frac{1}{\rho}(1-\rho)\,. \tag{3.26}$$

By inspection one finds that the on-shell action (3.2) evaluated on this background is divergent, and that only the first two terms in the expansion of W contribute to the IR divergences. To cancel them we add the counterterms

$$S_{\text{ct,bkgd}} = \int \mathrm{d}^4x\sqrt{\gamma}\left[\frac{3}{2} + \frac{1}{2}\Phi^2\right]. \tag{3.27}$$

With the addition of these terms the on-shell action is finite, but not zero because the Φ^4 term has a finite limit. The subtraction of finite terms corresponds to a choice of scheme. We proceed by subtracting the finite term so that the on-shell value of the action is zero when evaluated on the background, as required by supersymmetry. In other words, we supplement the counterterms in (3.27) by the finite counterterm

$$S_{\rm ct,fin} = \int d^4x\sqrt{\gamma}\frac{1}{18}\Phi^4\,. \tag{3.28}$$

We now proceed to obtain the general counterterms. For a scalar of AdS mass $m^2 = -3$, dual to an operator of conformal dimension $\Delta = 3$, the asymptotic expansion is

$$\Phi(x,\rho) = \rho^{1/2}[\phi_{(0)} + \rho(\phi_{(2)} + \log\rho\,\psi_{(2)}) + \cdots]\,. \tag{3.29}$$

The asymptotic solution can be found in the appendix A. The coefficient $\phi_{(2)}$ is undetermined and only the trace and divergence of $g_{(4)}$ are determined. The counterterms needed in order to cancel all divergences are given by

$$S_{ct}=\int_{\rho=\epsilon} \mathrm{d}^4x\sqrt{\gamma}\left(\frac{3}{2}-\frac{1}{8}R+\frac{1}{2}\Phi^2-\log\epsilon\left[\frac{1}{32}\left(R_{ij}R^{ij}-\frac{1}{3}R^2\right)+\frac{1}{4}\left(\Phi\Box_\gamma\Phi+\frac{1}{6}R\Phi^2\right)\right]\right), \tag{3.30}$$

where $\Box_\gamma$ is the laplacian of γ. These counterterms contain (3.27), as they should. We further supplement them with the finite counterterm in (3.28). This corresponds to choosing a supersymmetric scheme.

The renormalized action is defined

$$S_{\rm ren} = \lim_{\epsilon\to 0}[S_{\rm reg} + S_{\rm ct} + S_{\rm ct,fin}]\,. \tag{3.31}$$

The holographic one-point functions are equal to

$$\langle\mathcal{O}\rangle = \frac{1}{\sqrt{g_{(0)}}}\frac{\delta S_{\rm ren}}{\delta\phi_{(0)}} = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{3/2}}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\rm ren}}{\delta\Phi(x,\epsilon)}\right) = -2(\phi_{(2)} + \psi_{(2)}) + \frac{2}{9}\phi_{(0)}^3 \tag{3.32}$$

and

$$\begin{aligned}\langle T_{ij}\rangle = {} & g_{(4)ij} + \frac{1}{8}[\mathrm{Tr}\, g_{(2)}^2 - (\mathrm{Tr}\, g_{(2)})^2]\, g_{(0)ij} - \frac{1}{2}(g_{(2)}^2)_{ij} + \\ & + \frac{1}{4}\, g_{(2)ij}\,\mathrm{Tr}\, g_{(2)} + \left[\phi_{(0)}(\phi_{(2)} - \frac{1}{2}\psi_{(2)}) - \frac{1}{4}(\nabla\phi_{(0)})^2\right] g_{(0)ij} + \\ & + \frac{1}{2}\nabla_i\phi_{(0)}\nabla_j\phi_{(0)} + \frac{3}{4}(T^a_{ij} + T^\phi_{ij})\end{aligned} \tag{3.33}$$

where $T^a_{ij} + T^\phi_{ij}$ is the stress energy tensor of the the action equal to the integral of the trace anomaly $\mathcal{A} = \mathcal{A}_{grav} + \mathcal{A}_{scal}$ (see appendix A.2). The gravitational anomaly is given in (3.22) and the matter anomaly in (3.39) below. Although these terms are scheme dependent and can be omitted, we prefer to work with (3.33) keeping all the terms.

Under the transformation (3.15) the solution transforms as follows

$$\begin{aligned}
\phi'_{(0)}(x') &= \mu\phi_{(0)}(x'\mu) \\
\phi'_{(2)}(x') &= \mu^3[\phi_{(2)}(x'\mu) + \log\mu^2\psi_{(2)}(x'\mu)] \\
g'_{(0)}(x') &= g_{(0)}(x'\mu)\,, \qquad g'_{(2)}(x') = \mu^2 g_{(2)}(x'\mu) \\
g'_{(4)}(x') &= \mu^4\left[g_{(4)} + \log\mu^2\left(\frac{1}{2}T^a_{ij} + \frac{1}{2}T^\phi_{ij} - \phi_{(0)}\psi_{(2)}g_{(0)}\right)\right](x'\mu).
\end{aligned} \tag{3.34}$$

It follows that

$$\begin{aligned}
\langle\mathcal{O}(x')\rangle' &= \mu^3\left(\langle\mathcal{O}(x'\mu)\rangle - 2\log\mu^2\psi_{(2)}(x'\mu)\right) \\
\langle T_{ij}(x')\rangle &= \mu^4\left(\langle T_{ij}(x'\mu)\rangle + \frac{1}{2}\log\mu^2(T^a_{ij} + T^\phi_{ij})(x'\mu)\right).
\end{aligned} \tag{3.35}$$

Exactly as in the CB case, the new terms can be obtained from a local finite counterterm equal to $\mathcal{A}/2$, i.e. (3.18) but with $\mathcal{A} = \mathcal{A}_{\rm grav} + \mathcal{A}_{\rm scal}$ ($\mathcal{A}_{\rm grav}$ and $\mathcal{A}_{\rm scal}$ given in (3.22) and (3.39), respectively).

Evaluating these expressions on the background one obtains

$$\langle T_{ij}\rangle_B = 0\,, \qquad \langle\mathcal{O}\rangle_B = 0\,. \tag{3.36}$$

The first of these was guaranteed because we added $S_{\rm ct,fin}$ to enforce supersymmetry, but this same addition was crucial to obtain $\langle O\rangle_B = 0$. The latter is required by the physical interpretation that the $GPPZ$ flow at $\sigma = 0$ corresponds to a lagrangian deformation of $\mathcal{N} = 4$ SYM without vev.

Using the solution of the bulk field equations given in appendix A.2 one shows that

$$\nabla^i\langle T_{ij}\rangle = -\langle\mathcal{O}\rangle\nabla_j\phi_{(0)} \tag{3.37}$$

$$\langle T^i_i\rangle = -\phi_{(0)}\langle\mathcal{O}\rangle + \frac{1}{16}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) - \frac{1}{2}\left[(\nabla\phi_{(0)})^2 - \frac{1}{6}R\phi^2_{(0)}\right]\,. \tag{3.38}$$

These are the expected Ward identities. The second term in (3.38), is the holographic gravitational conformal anomaly (3.22) [28] and the last term,

$$\mathcal{A}_{\rm scal} = -\frac{1}{2}\left[(\nabla\phi_{(0)})^2 - \frac{1}{6}R\phi^2_{(0)}\right] \tag{3.39}$$

is a conformal anomaly due to matter [31]. The integrated anomaly should itself be conformal invariant, and indeed (3.39) is equal to the lagrangian of a conformal scalar. The coefficients are also the ones dictated by non-renormalization theorems.

4. Linearized analysis around domain-walls

As discussed in the Introduction we must go beyond the near boundary analysis to fix the undetermined coefficients and obtain correlation functions. A solution of the non-linear EOM's is beyond reach, but for our purpose of computing two-point functions it is sufficient to consider the linearized problem.

We look for fluctuations around the solution $(A(r), \varphi_B(r))$ of (2.3). The background plus linear fluctuations is described by

$$\begin{aligned} ds^2 &= e^{2A(r)}[\delta_{ij} + h_{ij}(x,r)]dx^i dx^j + (1+h_{rr})dr^2 \\ \Phi &= \varphi_B(r) + \tilde{\varphi}(x,r)\,, \end{aligned} \tag{4.1}$$

where h_{ij}, h_{rr} and $\tilde{\varphi}$ are considered infinitesimal. This choice does not completely fix the bulk diffeomorpisms. One can perform the one-parameter family of 'gauge transformations'

$$r = r' + \epsilon^r(r', x')\,, \qquad x^i = x'^i + \epsilon^i(r', x') \tag{4.2}$$

with

$$\epsilon^i = \delta^{ij} \int_r^\infty dr' e^{-2A(r)} \partial_j \epsilon^r\,, \tag{4.3}$$

where we only display the fluctuation-independent part of ϵ^i. These diffeos are related to those which induce the Weyl transformation (2.17) of the sources. The gauge choice is also left invariant by the linearization of the $4d$ diffeomorphisms in (2.16).

We have seen in section 2 that the $h^L_{(0)ij}$ part of $g_{(0)ij}$ drops out from the variation of the effective action. Equivalently the $4d$ diffeos can be fixed to set $h^L_{(0)ij} = 0$, see (2.16). We thus decompose the metric fluctuation as

$$h_{ij}(x,r) = h^T_{ij}(x,r) + \delta_{ij}\frac{1}{4}h(x,r) - \partial_i\partial_j H(x,r)\,. \tag{4.4}$$

Near the boundary the fluctuations $h^T_{ij}(x,r), h(x,r)$ and $H(x,r)$ admit an expansion similar to $g(x,\rho)$. The field theory sources are the leading ρ-independent parts. Analogously the scalar fluctuation $\tilde{\varphi}$ has the expansion given in (3.8) or (3.29) with source $\varphi_{(0)}(x)$.[12]

The equation for the transverse traceless modes decouples from the equations for $(\tilde{\varphi}, h, H, h_{rr})$. The coupled graviton-scalar field equations in the axial gauge where $h_{rr} = 0$ were derived in [4], and we now include h_{rr}. The fluctuation equations are

$$[\partial_r^2 + 4A'\partial_r + e^{-2A}\Box]f(x,r) = 0\,, \qquad h^T_{ij} = h^T_{(0)ij} f(x,r) \tag{4.5}$$

$$h' = -\frac{16}{3}\varphi'\tilde{\varphi} + 4A' h_{rr} \tag{4.6}$$

[12] A few words about our notation: as in (3.8) and (3.29), φ_B and $\tilde{\varphi}$ have a ρ-expansion. The corresponding coefficients will be denoted by $\varphi_{B(2k)}$ and $\varphi_{(2k)}$, $k = 0, 1, \ldots$. Then the coefficients appearing in the near-boundary analysis are given by $\phi_{(2k)} = \varphi_{B(2k)} + \varphi_{(2k)}$.

$$H'' + 4A'H' - \frac{1}{2}e^{-2A}h - h_{rr}e^{-2A} = 0 \tag{4.7}$$

$$2A'H' = \frac{1}{2}e^{-2A}h + \frac{8}{3}\frac{1}{p^2}W_\varphi(\tilde{\varphi}' - W_{\varphi\varphi}\tilde{\varphi} - \frac{1}{2}W_\varphi h_{rr})\,, \tag{4.8}$$

where $h^T_{(0)ij}$ is transverse, traceless, and independent of r.

The fluctuations $(\tilde{\varphi}, h, H, h_{rr})$ transform under the 'gauge transformations' in (4.2). One can form the following gauge invariant combinations

$$R \equiv h_{rr} - 2\partial_r\left(\frac{\tilde{\varphi}}{W_\varphi}\right), \qquad h + \frac{16}{3}\frac{W}{W_\varphi}\tilde{\varphi}\,, \qquad H' - \frac{2}{W_\varphi}e^{-2A}\tilde{\varphi}\,. \tag{4.9}$$

In terms of these variables the equations simplify,

$$h + \frac{16}{3}\frac{W}{W_\varphi}\tilde{\varphi} = -\frac{16}{3}\frac{e^{2A}}{p^2}\left(R(WW_{\varphi\varphi} - \frac{4}{3}W^2 - \frac{1}{2}W_\varphi^2) + \frac{1}{2}R'W\right) \tag{4.10}$$

$$H' - \frac{2}{W_\varphi}e^{-2A}\tilde{\varphi} = \frac{1}{p^2}\left(2R(W_{\varphi\varphi} - \frac{4}{3}W) + R'\right). \tag{4.11}$$

Equation (4.6) takes the form

$$\left(h + \frac{16}{3}\frac{W}{W_\varphi}\tilde{\varphi}\right)' = -\frac{8}{3}WR\,. \tag{4.12}$$

Differentiating (4.10) leads to the second order differential equation

$$R'' + (2W_{\varphi\varphi} - 4W)R' - \left(4W_\varphi^2 - 2W_\varphi W_{\varphi\varphi\varphi} - \frac{32}{9}W^2 + \frac{8}{3}WW_{\varphi\varphi} + p^2e^{-2A}\right)R = 0\,. \tag{4.13}$$

Equations (4.10)–(4.12) are invariant under the transformations (4.2), and this is important for the success of the present method. We will use these equations, but henceforth consider the theory in the axial gauge, i.e. we set $h_{rr} = 0$. In [4] the axial gauge was imposed early. The resulting equations were equivalent to (4.10)–(4.12) but were processed differently. In particular because of the assumed conditions at the interior singularity, the boundary data for metric and field sources were not independent, and (at least in the GPPZ case) they were even non-locally related. This made it difficult to obtain 2-point functions, as one would have to disentangle the non-local relations of the sources from the true non-local 2-point function.

5. Correlation functions

5.1 Coulomb branch flow

Let us now discuss the two-point functions for the trace-scalar sector. The differential equation for R becomes,

$$R''(v) + \frac{2}{v}R'(v) - \frac{p^2}{4l^2}\frac{1}{v^2(1-v)}R(v) = 0\,, \tag{5.1}$$

where the prime indicates derivative with respect to v. The solution of this equation is

$$R(v,p) = v^a(1-v)F(1+a, 2+a, 2+2a; v)\,, \tag{5.2}$$

where $a = -\frac{1}{2}(1 - \sqrt{1+\frac{p^2}{l^2}})$. Equations (4.10) become

$$h(v) = \frac{4}{3}(v+2)\left(\frac{\sqrt{6}}{(1-v)}\tilde{\varphi} + \frac{2l^2}{p^2}vR'\right) + \frac{16l^2}{3p^2}R \tag{5.3}$$

$$H'(v) = \frac{1}{2v}\left(\frac{\sqrt{6}}{(1-v)}\tilde{\varphi} + \frac{2l^2}{p^2}vR'\right) + \frac{2l^2}{3vp^2}R \tag{5.4}$$

One can solve (5.3) for $\tilde{\varphi}_{(0)}$ (the coefficient of order ρ in the near-boundary expansion of $\tilde{\varphi}(x,\rho)$),

$$\tilde{\varphi}_{(0)} = \frac{1}{4\sqrt{6}}h_{(0)} + \varphi_{(0)}[-\bar{Q} + \frac{l^2}{p^2}(4a^2 - \frac{8}{3})]\,, \tag{5.5}$$

where $\bar{Q} = \psi(1) + \psi(2) - \psi(1+a) - \psi(2+a)$ [32]. It follows that

$$\langle\mathcal{O}\rangle = \frac{N^2}{2\pi^2}\left(-\frac{2l^2}{\sqrt{6}} + \frac{1}{2\sqrt{6}}h_{(0)} + \varphi_{(0)}\left[4\psi(1+a) - 4\psi(1) - \frac{16l^2}{3p^2}\right]\right). \tag{5.6}$$

Substituting this relation in (2.23), using $\sqrt{g_{(0)}} = 1 + \frac{1}{2}h_{(0)} + \frac{1}{2}p^2H_{(0)}$, and going to momentum space one finds that (2.23) can be easily integrated to

$$\begin{aligned} S_{\rm ren}[h_{(0)}, H_{(0)}, \varphi_{(0)}] = \frac{N^2}{2\pi^2}\int d^4p\Big(&-\frac{1}{2\sqrt{6}}\varphi_{(0)}h_{(0)} - \frac{p^2}{\sqrt{6}}\varphi_{(0)}H_{(0)} + \varphi_{(0)}\times \\ &\times\left[-\frac{2l^2}{\sqrt{6}} + \varphi_{(0)}\left(2\psi(1+a) - 2\psi(1) - \frac{8l^2}{3p^2}\right)\right]\Big). \end{aligned} \tag{5.7}$$

From here one immediately reads all two-point functions,

$$\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle = \frac{N^2}{2\pi^2}\left[4\psi(1) - 4\psi(1+a) + \frac{16l^2}{3p^2}\right] \tag{5.8}$$

$$\langle T_i^i(p)\mathcal{O}(-p)\rangle = -\frac{4l^2}{\sqrt{6}}\frac{N^2}{2\pi^2} = 2\langle\mathcal{O}\rangle_B \tag{5.9}$$

$$p^ip^j\langle T_{ij}(p)\mathcal{O}(-p)\rangle = -\frac{2l^2}{\sqrt{6}}\frac{N^2}{2\pi^2}p^2 = \langle\mathcal{O}\rangle_B p^2 \tag{5.10}$$

These correlation functions, as well as the one in (5.13), correctly approach the corresponding AdS correlators, including normalizations, in the limit of vanishing vev, i.e. $l \to 0$. Notice that the contact term in (5.9) and (5.10) are exactly the ones dictated by the Ward identities (3.21) and (3.20).

It remains to discuss the tranverse traceless sector. To determine the two-point function we need to obtain $g_{(4)ij}$ as a function of $h^T_{(0)ij}$. The solution of the fluctuation equation (4.5) is given by [6]

$$f(v,p) = v^aF(a, a, 2+2a; v)\,. \tag{5.11}$$

Expanding around $v = 1$ and converting to the ρ variable one obtains

$$h^T_{(4)ij} = h^T_{(0)ij}\left(\frac{p^4}{32}[\psi(1)+\psi(3)] + \frac{p^2 l^2}{24} - \frac{p^4}{16}\psi(a+1)\right). \tag{5.12}$$

The first term on the right hand side yields only a contact term and can be omitted when computing correlators at non-coincident points. From the linearization of (3.13) we obtain the correlation function

$$\langle T_{ij}(p) T_{kl}(-p)\rangle = \P^{TT}_{ijkl}\frac{N^2}{2\pi^2}\left(\frac{p^2 l^2}{12} - \frac{p^4}{8}\psi(a+1)\right). \tag{5.13}$$

Both (5.8) and (5.13) contain 0-mass poles. In (5.13) the projection operator gives the pole term $p_i p_j p_k p_l / p^2$. These poles have physical sign. Their contribution dominates the long-distance behavior of the Euclidean correlation functions and obeys reflection positivity. The poles are a manifestation of the expected dilaton, the Goldstone boson of spontaneously broken conformal symmetry. Pole residues are proportional to the vev $\langle \mathcal{O}\rangle_B$. In addition to the massless poles, there is a branch cut along the positive real axis indicating a continuous spectrum above the mass scale

$$m_W = \frac{l}{\alpha'} \tag{5.14}$$

in exact agreement with gauge theory expectations: l/α' is the average mass of the W-bosons [6], and the vev that breaks the symmetry is proportional to l^2. The connected correlator (see Footnote 10)

$$\langle T_{ij}(p)\mathcal{O}(-p)\rangle = \frac{4l^2}{3\sqrt{6}}\frac{N^2}{2\pi^2}\pi_{ij} = -\frac{2}{3}\langle\mathcal{O}\rangle_B \pi_{ij} \tag{5.15}$$

has the dilaton pole but is otherwise purely local. Furthemore,

$$\langle T^i_i(p) T^j_j(-p)\rangle = 0 \tag{5.16}$$

as can be seen from (5.7) or from the linearization of (3.21). These new results from supergravity are fully consistent with spontaneously broken conformal symmetry in the dual field theory!

5.2 *GPPZ* flow

The equation for R becomes

$$R'' + \frac{1}{u(1-u)}\left[R' + \left(\frac{2u-1}{u(1-u)} - \frac{p^2}{4}\right)R\right] = 0\,, \tag{5.17}$$

where the prime indicates differentiation with respect to u. The solution of this equation is

$$R(u,p) = u(1-u)F\left(\frac{3}{2}+\frac{1}{2}b, \frac{3}{2}-\frac{1}{2}b, 3; u\right), \tag{5.18}$$

where $b = \sqrt{1-p^2}$.

Using the definition of R in (4.9) we find

$$\varphi_{(2)} + \psi_{(2)} = \varphi_{(0)} - p^2 \left(\frac{1}{4}\varphi_{(0)} + \frac{\sqrt{3}}{32} h_{(0)} \right) \bar{J}\,, \tag{5.19}$$

where we used $\psi_{(2)} = p^2(\frac{1}{4}\varphi_{(0)} + \frac{\sqrt{3}}{32}h_{(0)})$ from (A.6) and $\bar{J} = 2\psi(1) - \psi(\frac{3}{2} + \frac{1}{2}b) - \psi(\frac{3}{2} - \frac{1}{2}b)$ from [32]. The linearization of (3.32) then gives

$$\langle \mathcal{O} \rangle = -2(\varphi_{(2)} + \psi_{(2)}) + \frac{2}{3}\varphi_B^2 \varphi_{(0)} = p^2 \left(\frac{1}{2}\varphi_{(0)} + \frac{\sqrt{3}}{16} h_{(0)} \right) \bar{J}\,. \tag{5.20}$$

Inserting $\langle \mathcal{O} \rangle$ in (2.23) we obtain the action

$$S_{\rm ren}[h_{(0)}, H_{(0)}, \varphi_{(0)}] = \frac{N^2}{2\pi^2} \int d^4p \left(\frac{1}{4}(\varphi_{(0)} + \frac{\sqrt{3}}{8}h_{(0)})^2 p^2 \bar{J} + \frac{3}{256} p^2 h_{(0)}^2 \right), \tag{5.21}$$

where we have also included the overall factor $N^2/2\pi^2$. This action gives all correlators of T_i^i and $\mathcal{O}$. The last term is due to the anomaly. It follows that there is the operator relation,

$$T_i^i = \beta_{\mathcal{O}} \mathcal{O}, \tag{5.22}$$

where $\beta_{\mathcal{O}} = -\varphi_{(0)B} = -\sqrt{3}$. We will discuss this below.

The correlation functions are equal to

$$\begin{aligned}
\langle \mathcal{O}(p)\mathcal{O}(-p) \rangle &= -\frac{1}{2}\frac{N^2}{2\pi^2}p^2 \bar{J} && (5.23)\\
\langle T_i^i(p)\mathcal{O}(-p) \rangle &= \frac{\sqrt{3}}{2}p^2 \frac{N^2}{2\pi^2}\bar{J} && (5.24)\\
\langle T_i^i(p)T_i^i(-p) \rangle &= -\frac{3}{2}\frac{N^2}{2\pi^2}p^2(\bar{J}+1) && (5.25)
\end{aligned}$$

The correlator $\langle \mathcal{O}(p)\mathcal{O}(-p) \rangle$ agrees with [8]. The shift from $\bar{J}$ to $\bar{J}+1$ in (5.25) is due to the linearization of the trace anomaly term in (3.38). There is then a cancellation, and the correlator vanishes at the rate p^4 at low momentum. This is relevant to the issue of 0-mass poles discussed below.

We now turn to the transverse traceless correlator. The solution to the fluctuation equation is [9, 4]

$$h_{ij}^T(u,p) \propto (1-u)^2 F\left(2 + \frac{ip}{2}, 2 - \frac{ip}{2}, 2; u\right) h_{(0)ij}^T(p) \tag{5.26}$$

which has the asymptotic expansion [32]

$$h_{ij}^T = \left[1 - \rho\frac{p^2}{4} + \rho^2 \frac{1}{32}p^2(4+p^2)(\bar{K} - \log\rho) + \cdots\right] h_{(0)ij}^T, \tag{5.27}$$

where $\bar{K} = \psi(1) + \psi(3) - \psi(2 + \frac{ip}{2}) - \psi(2 - \frac{ip}{2})$. After careful linearization of (3.33) one finds the following contribution to the transverse traceless correlator

$$\langle T_{ij} \rangle = g_{(4)ij} - \frac{3}{16} p^2 h^T_{(0)ij} = h_{(4)ij} - h_{(2)ij} - \frac{3}{16} p^2 h^T_{(0)ij} \, , \tag{5.28}$$

where $h_{(2)}$ and $h_{(4)}$ are the order ρ and ρ^2 terms in the expansion (5.27). The first term in (5.28) is the linearization of $g_{(4)ij}$ in (3.33); the second is a linear correction to $g_{(4)ij}$ arising because the background scale factor in (3.26) has an order ρ contribution; finally the last term comes from the linearization of the term $3T^{\phi}_{ij}/4$ in (3.33). The desired correlation function is then

$$\langle T_{ij}(p) T_{kl}(-p) \rangle_{TT} = \P^{TT}_{ijkl} \frac{N^2}{2\pi^2} \left[\frac{1}{16} p^2 (p^2 + 4) \bar{K} + \frac{p^2}{8} \right] . \tag{5.29}$$

To this one must add the trace part obtained from (5.25)

$$\langle T_{ij}(p) T_{kl}(-p) \rangle_{\mathrm{trace}} = -\pi_{ij} \pi_{kl} \frac{N^2}{2\pi^2} p^2 \frac{(\bar{J} + 1)}{6} \, . \tag{5.30}$$

We now discuss some of the physical aspects of our results. First we note that the correlators themselves are not completely new, [9, 8, 10]. Rather, it is the coherent derivation via the holographic renormalization procedure which is to be emphasized.

Physically one should expect that correlators become insensitive to the shape of the domain wall as $p^2 \to \infty$ and have the same limiting form $p^{2\Delta - 4} \log p$ as those in a pure AdS_5 background. This property is satisfied by all CB and $GPPZ$ correlators in this paper. With due care for conventions, this limit may be used to normalize all results.

The $\beta_{\mathcal{O}}$ function which appears is a trivial constant. In fact since $\varphi_{(0)B} = \sqrt{3}$, it agrees with the classical value[13] in (2.19). This is exactly what we expect here, since the $GPPZ$ flow is dual to the deformation of $\mathcal{N} = 4$ SYM superpotential by the $\mathcal{N} = 1$ supersymmetric mass terms $W = m \S_i \mathrm{Tr}(\Phi_i^2)$ with $m = \varphi_{(0)B}$. The $\mathcal{N} = 1$ non-renormalization theorem implies that m is renormalized only through anomalous dimension of the operator it multiplies. However, in our case W is a protected operator in the undeformed theory, and there is no anomalous dimension.[14] The $\beta_{\mathcal{O}}$ function is indeed just classical. Notice that our $\beta_{\mathcal{O}}$ function is equal to the leading term in the ρ expansion of the "holographic" beta function of [36, 37].

In general our correlation functions have the discrete spectrum of poles noted in earlier discussions of the $GPPZ$ flow. However there is a more delicate question of a

[13] See also [33, sec. 12.5].

[14] Contrary to naive expectation, the supersymmetric mass term does not involve the lowest scalar operator $\mathcal{K} = \S_i \mathrm{Tr}(\phi_i^2)$ in the $\mathcal{N} = 4$ Konishi multiplet [34], with non-zero anomalous dimension at weak coupling [34, 35] that grows as $(g_s N)^{1/4}$ at strong coupling. Indeed the Konishi scalar is the simplest among the operators dual to string excitations [2, 34] not captured by the supergravity limit.

0-mass pole. One might have naively expected that terms in the invariant amplitudes $A(p^2)$ and $B(p^2)$ that vanish in the limit $p^2 \to 0$ are just contact terms. However, because the correlators involve projection operators, terms that vanish at the rate p^2 yield zero-mass poles instead. The presence of such a zero-mass pole would conflict with the interpretation that the $GPPZ$ flow is dual at long distance to $\mathcal{N} = 1$ SYM which is a confining theory with mass gap. We find that both $A(p^2)$ and $B(p^2)$ vanish as p^4 at low momentum, and thus the correlators do not contain zero-mass poles.

One may ask whether our results depend on the specific radial coordinate used, specifically the scaling $\rho \to \mu^2\rho$ associated with an RG transformation in section 3. It was shown there that the rescaling introduces new terms in the stress-energy tensor which can be derived from the anomaly action which is local and thus produces only contact terms in correlation functions such as the complete $\langle T_{ij}T_{kl}\rangle$. Since our method computes the TT and trace parts of $\langle T_{ij}T_{kl}\rangle$ separately one can see from hypergeometric series such as (5.27) that terms of order $p^2 \log \mu^2$ appear. These potentially give 0-mass poles. However, one can check explicitly that the 0-mass poles cancel between the TT and trace parts of the correlator, leaving only the net $p^4 \log \mu^2$ contact term from the anomaly.

6. Conclusion

In this paper we have developed a coherent approach to correlation functions of the stress tensor in holographic RG flows. The implementation of holographic renormalization is somewhat tedious, but valid for all solutions of a given bulk action. The procedure gives relatively simple finite expressions for source dependent vevs $\langle T_{ij}\rangle$ and $\langle \mathcal{O}\rangle$ from which correlation functions can easily be obtained. Ward identities with correct anomalies are satisfied. The expected physics of flows describing both operator and Coulomb deformations of $\mathcal{N} = 4$ SYM is reflected in the results. There is more to be done. The correlation functions presumably satisfy Callan-Symanzik equations, see [38] for a recent discussion. The treatment should be extended to other important field theory operators such as vector currents. We hope to discuss these questions and present more of the technical details of the procedure in [17].

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A. Asymptotic solutions

A.1 Coulomb branch flow

Scalar field:

$$\begin{aligned}
\phi_{(2)} &= -\frac{1}{4}\left(\Box_0\phi_{(0)} + \frac{2}{3}\phi_{(0)}R[g_{(0)}]\right) - \frac{4}{\sqrt{6}}\left(\phi_{(0)}^2 - \frac{1}{2}\phi_{(0)}\tilde{\phi}_{(0)}\right) \\
\tilde{\phi}_{(2)} &= -\frac{1}{4}\left(\Box_0\tilde{\phi}_{(0)} + \frac{1}{3}R[g_{(0)}](\tilde{\phi}_{(0)} + \phi_{(0)}) + 8(\phi_{(2)} + \psi_{(2)})\right) + \frac{1}{\sqrt{6}}\tilde{\phi}_{(0)}^2 \\
\psi_{(2)} &= \frac{1}{\sqrt{6}}\phi_{(0)}^2 \,.
\end{aligned} \tag{A.1}$$

Metric:

$$\begin{aligned}
g_{(2)ij} &= \frac{1}{2}\left(R_{ij} - \frac{1}{6}R\,g_{(0)ij}\right), \\
h_{1(4)ij} &= h_{(4)ij} - \frac{2}{3}\phi_{(0)}\tilde{\phi}_{(0)}g_{(0)ij}\,, \qquad h_{2(4)ij} = -\frac{1}{3}\phi_{(0)}^2 g_{(0)ij} \\
\mathrm{Tr}g_{(4)} &= \frac{1}{4}\mathrm{Tr}g_{(2)}^2 - \frac{2}{3}(\phi_{(0)}^2 + 2\tilde{\phi}_{(0)}^2)\,, \\
\nabla^j g_{(4)ij} &= \nabla^j\left(-\frac{1}{8}[\mathrm{Tr}\,g_{(2)}^2 - (\mathrm{Tr}\,g_{(2)})^2]\,g_{(0)ij} + \frac{1}{2}(g_{(2)}^2)_{ij} - \right. \\
&\qquad \left. - \frac{1}{4}\,g_{(2)ij}\,\mathrm{Tr}\,g_{(2)} - \frac{1}{3}(2\phi_{(0)}^2 + \tilde{\phi}_{(0)}^2 - 3\phi_{(0)}\tilde{\phi}_{(0)})g_{(0)ij}\right) - \\
&\qquad -2\tilde{\phi}_{(0)}\nabla_i\phi_{(0)}\,.
\end{aligned} \tag{A.2}$$

The tensor $h_{(4)}$ is equal to [1]

$$\begin{aligned}
h_{(4)} &= \frac{1}{8}R_{ikjl}R^{kl} + \frac{1}{48}\nabla_i\nabla_j R - \frac{1}{16}\nabla^2 R_{ij} - \frac{1}{24}RR_{ij} + \\
&\quad + \left(\frac{1}{96}\nabla^2 R + \frac{1}{96}R^2 - \frac{1}{32}R_{kl}R^{kl}\right) \\
&= \frac{1}{2}T^a_{ij}\,,
\end{aligned} \tag{A.3}$$

where T^a_{ij} is the stress energy tensor derived from the action

$$S_a = \int \mathrm{d}^4x\sqrt{g_{(0)}}\mathcal{A}_{\mathrm{grav}}\,. \tag{A.4}$$

$\mathcal{A}_{grav}$ is the gravitaional trace anomaly given in (3.22).

A.2 *GPPZ* flow

Scalar field:

$$\psi_{(2)} = -\frac{1}{4}\left(\Box_0\phi_{(0)} + \frac{1}{6}R\phi_{(0)}\right). \tag{A.5}$$

Metric:

$$\begin{aligned}
g_{(2)ij} &= \frac{1}{2}\left(R_{ij} - \frac{1}{6}R\,g_{(0)ij}\right) - \frac{1}{3}\phi_{(0)}^2 g_{(0)ij} \\
h_{1(4)ij} &= h_{(4)ij} + \frac{1}{12}R_{ij}\phi_{(0)}^2 - \frac{1}{3}\nabla_i\phi_{(0)}\nabla_j\phi_{(0)} + \frac{1}{12}(\nabla\phi_{(0)})^2 g_{(0)ij} + \\
&\quad + \frac{1}{6}\phi_{(0)}\nabla_i\nabla_j\phi_{(0)} + \frac{1}{12}\phi_{(0)}\Box_0\phi_{(0)}g_{(0)ij} \\
&= h_{(4)ij} + \frac{1}{2}T^{\phi}_{ij} + \frac{1}{4}g_{(0)ij}\left(\phi_{(0)}\Box_0\phi_{(0)} + \frac{1}{6}R\phi_{(0)}^2\right) \\
h_{2(4)ij} &= 0\,, \\
\mathrm{Tr} g_{(4)} &= -2\phi_{(0)}\phi_{(2)} - \frac{1}{4}\phi_{(0)}\Box_0\phi_{(0)} - \frac{5}{72}R\phi_{(0)}^2 + \\
&\quad + \frac{1}{16}\left(R_{ij}R^{ij} - \frac{2}{9}R^2\right) + \frac{2}{9}\phi_{(0)}^4 \\
\nabla^j g_{(4)ij} &= \nabla^j\Bigg(-\frac{1}{8}[\mathrm{Tr}\, g_{(2)}^2 - (\mathrm{Tr}\, g_{(2)})^2]\, g_{(0)ij} + \frac{1}{2}(g_{(2)}^2)_{ij} - \\
&\qquad - \frac{1}{4}\, g_{(2)ij}\,\mathrm{Tr}\, g_{(2)} - \frac{3}{2}h_{1(4)ij} - \frac{1}{2}\nabla_i\phi_{(0)}\nabla_j\phi_{(0)} + \\
&\qquad + g_{(0)ij}\left[\frac{1}{4}(\nabla\phi_{(0)})^2 - \phi_{(0)}(\phi_{(2)} + \psi_{(2)})\right]\Bigg) - \\
&\quad - \left[-2(\phi_{(2)} + \psi_{(2)}) + \frac{2}{9}\phi_{(0)}^3\right]\nabla_i\phi_{(0)}
\end{aligned} \tag{A.6}$$

where T^{ϕ}_{ij} is the stress energy tensor derived from the action

$$S_\phi = \int \mathrm{d}^4x\sqrt{g_{(0)}}\mathcal{A}_{\mathrm{scal}}\,. \tag{A.7}$$

$\mathcal{A}_{\mathrm{scal}}$ is the matter conformal given in (3.39). $h_{(4)}$ is given by the same formula (A.3) as in the CB case.

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